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Bisimulation for Communicating Piecewise Deterministic Markov Processes (CPDPs)

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Abstract. CPDPs (Communicating Piecewise Deterministic Markov Processes) can be used for compositional specification of systems from the class of stochastic hybrid processes formed by PDPs (Piecewise Deterministic Markov Processes). We define CPDPs and the composition of CPDPs, and prove that the class of CPDPs is closed under composition. Then we introduce a notion of bisimulation for PDPs and CPDPs and we prove that bisimilar PDPs as well as bisimilar CPDPs have equal stochastic behavior. Finally, as main result, we prove the congruence property that, for a composite CPDP, substituting components by different but bisimilar components results in a CPDP that is bisimilar to the original composite CPDP (and therefore has equal stochastic behavior).

1 Introduction

Many real-life systems nowadays are complex hybrid systems. They consist of multiple components 'running' simultaneously, having both continuous and discrete dynamics and interacting with each other. Also, many of these systems have a stochastic nature. An interesting class of stochastic hybrid systems is formed by the Piecewise Deterministic Markov Processes (PDPs), which were introduced in 1984 by Davis (see [1, 2]). Motivation for considering PDP systems is two-fold. First, almost all stochastic hybrid processes that do not include diffusions can be modelled as a PDP, and second, PDP processes have very nice properties (such as the strong Markov property) when it comes to stochastic analysis. (In [2] powerful analysis techniques for PDPs have been developed). However, PDPs cannot communicate or interact with other PDPs and therefore, from a compositional modelling point of view, we should find a way of opening the structure of PDPs to let them communicate/interact.

In [3], the automata formalism CPDP, which stands for Communicating Piecewise Deterministic Markov Processes, is introduced. Basically, a CPDP is a PDP-type system that can communicate (or interact) with other CPDPs. In [3], this communication is formalized by means of a composition operator. In this way, we may model complex stochastic hybrid systems (without diffusions) as PDPs, based on the description of their components. Furthermore, in [4], it

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is proven that for any CPDP that is closed, i.e. does not communicate anymore with the environment, we can construct a corresponding PDP that expresses the same stochastic process. Therefore, analysis techniques for PDPs can be used for analyzing CPDPs.

In this paper we give a slightly different definition of CPDPs than the definition in [3]. This new definition is more convenient in the context of composition. As in [3], we formalize the communication between CPDPs by means of a composition operator, and we prove that the composition of two CPDPs is again a CPDP. (A partial proof of this was already given in [3]).

The main part of this paper is about bisimulation for CPDPs. It is well-known that the composition of multiple subsystems leads to state space explosion. One tool that has proved to be effective in dealing with the state space explosion problem is bisimulation. Bisimulation can be seen as a state space reduction technique: By bisimulation we can find systems with smaller state spaces, that still have the same external behavior. Two systems have the same external behavior if they cannot be distinguished in any composition context. The notion of bisimulation was introduced by Milner [5] in the context of discrete state processes. Since then, bisimulation has also been established in the context of probabilistic and stochastic automata [6, 7], continuous time interactive Markov chains (IMC) [8], continuous dynamical systems [9, 10] and general (non-stochastic) hybrid systems [11, 12].

In this paper, we define bisimulation in the context of CPDPs. In some sense, this notion of bisimulation for CPDPs integrates the notions of bisimulation for IMC, stochastic automata and continuous/hybrid systems.

An important point is that CPDPs have a stochastic processes semantics (see [4]). This implies that we want to define bisimulation in such a way that two bisimilar CPDPs express equivalent stochastic processes. Therefore, we define bisimulation such that certain analytical properties of stochastic processes still remain in the quotient systems obtained by bisimulation (by factoring out equivalence classes). In particular, we prove that two bisimilar CPDPs have the same stochastic (PDP) behavior. We also prove the congruence property that, in the composition of multiple CPDPs, substitution of a component by a different bisimilar component does not change the stochastic behavior of the composite system.

From an analysis point of view, we can then reduce the state space of a composite CPDP in a compositional way by substituting components by state-reduced bisimilar components. To analyze the original composite CPDP, we can then (because of the equivalence result of CPDPs and PDPs) use the PDP analysis techniques on the state reduced composite CPDP.

The organization of the paper is as follows. In Section 2 we give the definition of the PDP stochastic process. In Section 3 we give the definition of the CPDP model. In Section 4 we define composition for CPDP and we prove that, under certain conditions, the composition of two CPDPs is again a CPDP. In Section 5 we prepare the bisimulation notion for CPDPs by first defining bisimulation for PDPs with output functions (called weighted PDPs). We prove that bisimilar weighted PDPs have equivalent stochastic behavior. Then in Section 6 we extend the PDP bisimulation notion to CPDPs. Using the results of Section 5, we prove that weighted bisimilar CPDPs have equivalent stochastic behavior. After that, we prove that, in the composition of multiple weighted CPDPs, substitution of a component by a different bisimilar component does not change the stochastic behavior of the composite system. In the final section conclusions are drawn and future research directions are discussed.

2 Definition of the PDP

The state space and the dynamics of a PDP are defined as follows: K is a countable set of locations. For each $\nu \in K$, $d(\nu) \in \mathbb{N}$ denotes the dimension of the continuous state space of location ν. For each $\nu \in K$, let E_ν be an open subset of $\mathbb{R}^{d(\nu)}$ and let $g_\nu : \mathbb{R}^{d(\nu)} \to \mathbb{R}^{d(\nu)}$ be a locally Lipschitz continuous function on E_ν. The flow $\phi_\nu(t,\zeta)$ is uniquely determined by the differential equation $\dot{\hat{\zeta}} = g_\nu(\hat{\zeta})$ and equals $\hat{\zeta}(t)$, assumed that $\hat{\zeta}(0) = \zeta$. The hybrid state space of the PDP is now defined as

$$E = \{(\nu,\zeta) | \nu \in K, \zeta \in E_\nu\}.$$

Remark 1. In fact, the state space E of the PDP is in [2] extended such that E also contains the boundary points that are backward reachable (via flow ϕ) but not forward reachable from the interior of E.

For $x = (\nu,\zeta) \in E$ define

$$t_*(x) = \begin{cases} \inf\{\mathrm{t} > 0 | \phi_\nu(\mathrm{t},\zeta) \in \partial \mathrm{E}_\nu\}, \\ \infty \text{ if no such time exists.} \end{cases}$$

where $\partial E_\nu = \bar{E}_\nu \backslash E_\nu$ is the boundary of E_ν, $\bar{E}_\nu$ is the closure of E_ν.

The jump mechanism of the PDP is determined by a jump rate function λ and a transition measure Q. The jump rate $\lambda : E \to \mathbb{R}_+$ is a measurable function such that for each $x = (\nu,\zeta) \in E$, there exists $\epsilon(x) > 0$ such that the function $s \to \lambda(\nu, \phi_\nu(s,\zeta))$ is integrable on $[0, \epsilon(x)[$. With Γ^* we denote the boundary of E that is reachable from the interior of E. The transition measure Q maps $E \cup \Gamma^*$ into the set $\mathcal{P}(E)$ of probability measures on the Borel space $(E, \mathcal{E})$, where $\mathcal{E}$ is the set containing all Borel sets of E (according to a 'natural' topology, defined in [2]), with the properties that for each fixed $A \in \mathcal{E}$ the map $x \to Q(A,x)$, where $Q(A,x)$ denotes the probability of A according to the probability measure $Q(x)$, is measurable, and $Q(\{x\}, x) = 0$ for all $x \in E \cup \Gamma^*$.

A PDP process, starting from initial state $x_0 = (\nu_0, \zeta_0)$, can be 'executed' as follows: The dynamics of x_t from $t = 0$ is determined by the vectorfield g_{ν_0} until either the boundary (i.e. the set ∂E_{ν_0}) is hit at time $t_*(x_0)$ or until a point is generated by the Poisson process that has density $\lambda(x_t)$. In either case, a jump takes place and the target hybrid state is determined by the probability measure $Q(\cdot,(\nu_0,\phi_{\nu_0}(\hat{t},\zeta_0)))$, where $\hat{t}$ is the jump time. From the target state this execution procedure can be repeated.

For a PDP it is assumed that there are no explosions (i.e. $|\phi_\nu(t,\zeta)| \not\to \infty$ if $t \not\to \infty$) and that there is no Zeno behavior (i.e. for every starting point $x \in E$, $\mathrm{E}N_t < \infty$ for all $t \in \mathbb{R}_+$, where N_t is a random variable 'counting' the number of jumps up to time t and $\mathrm{E}N_t$ is the expectation of N_t).

3 Definition of the CPDP

A CPDP automaton is a tuple $(L, V, v, Inv, G, \Sigma, A, P, S, C)$, where

- L is a countable set of locations
- V is a set of variables. With $d(y)$ for $y \in V$ we denote the dimension of variable y. $y \in V$ takes its values in $\mathbb{R}^{d(y)}$. We also say that $\mathbb{R}^{d(y)}$ is the valuation space of y.
- $v : L \to 2^V$ maps each location to a subset of V, which is the set of active variables of the corresponding location
- Inv assigns to each location l and each variable $y \in v(l)$ an open subset of $\mathbb{R}^{d(y)}$, i.e. $Inv(l,y) \subset \mathbb{R}^{d(y)}$. With Inv_l we denote the subset of the valuation space of $v(l)$ that is built from (or loosely speaking: is the product of) the invariants of the individual variables. With ∂Inv_l we denote the set of boundary points of l, which is equal to the set of valuations of $v(l)$ where each $y \in v(l)$ takes value in $\overline{Inv(l,y)}$ and at least one $y \in v(l)$ takes value in $\partial Inv(l,y) := \overline{Inv(l,y)} \backslash Inv(l,y)$.
- G assigns to each location l and each $y \in v(l)$ a locally Lipschitz continuous function from $\mathbb{R}^{d(y)}$ to $\mathbb{R}^{d(y)}$, i.e. $G(l,y) : \mathbb{R}^{d(y)} \to \mathbb{R}^{d(y)}$. This vectorfield uniquely determines a flow $\phi_{l,y}(t,y_0)$ along this vectorfield.
- Σ is the set of communication labels. $\bar{\Sigma}$ denotes the 'passive' mirror of Σ and is defined as $\bar{\Sigma} = \{\bar{a} | a \in \Sigma\}$.
- B is a finite set of boundary hit transitions and consists of 4-tuples (l,a,l',R), denoting a transition from location $l \in L$ to location $l' \in L$ with communication label $a \in \Sigma$ and reset map R. This reset map R assigns to each boundary point of l for each active variable $y \in v(l')$ a probability measure on the invariant (and its Borel sets) of y for location l'. We will denote the measure of R for variable y at boundary point ζ by $R^y(\zeta)$.
- P is a finite set of passive transitions and consists of 4-tuples $(l,\bar{a},l',R)$, denoting a transition from location $l \in L$ to location $l' \in L$ with passive communication label $\bar{a} \in \bar{\Sigma}$ and reset map R. R assigns to each interior point of location l for each active variable $y \in v(l')$ a probability measure on the invariant (and its Borel sets) of y for location l'.
- S is a finite set of spontaneous (also called Poisson) transitions and consists of 5-tuples (l,λ,a,l',R), denoting a transition from location $l \in L$ to location $l' \in L$ with communication label $a \in \Sigma$, jump-rate function λ and reset map R. The jump rate $\lambda : Inv_l \to \mathbb{R}_+$ is a measurable function such that for each $\zeta \in Inv_l$, there exists $\epsilon(\zeta) > 0$ such that the function $s \to \lambda(\phi_l(s,\zeta))$ is

integrable on $[0, \epsilon(\zeta)[$, where ϕ_l denotes the flow of the valuations of variables $v(l)$ for location l. R is defined on all interior points of l as it is done for passive transitions.

- C is the choice function. C assigns to each boundary point (l, ζ) of the CPDP a probability measure on the set of outgoing boundary hit transitions, i.e. $C(l, \zeta)$ (with $\zeta \in \partial Inv_l$) is a probability measure on B_l, where B_l is the set of boundary hit transitions that have l as origin location. Furthermore, for all $l \in L$ and all $\bar{a} \in \bar{\Sigma}$, such that for location l there is an outgoing passive transition labelled $\bar{a}$, C assigns to each triplet $(l, \zeta, \bar{a})$ (with $\zeta \in Inv_l$) a probability measure on the set of passive transitions leaving l and labelled $\bar{a}$.

We also impose the standard PDP conditions on a CPDP. For the details of how this is done, we refer to [4].

Passive transitions are used to interact with the environment (see [3] for an explanation of the communication mechanism established by the interplay of boundary hit, spontaneous and passive transitions). The environment can activate/trigger these passive transitions. When a CPDP does not have passive transitions, then it can not be influenced by the environment, which means that it is autonomous and can be executed 'on its own'.

Execution of a CPDP $(L, V, v, Inv, G, \Sigma, A, P, S, C)$ without passive transitions (i.e. $P = \emptyset$), starting from initial state $x_0 = (l_0, \zeta_0)$, is done as follows: The dynamics at $t = 0$ is determined by the vectorfield $G(l_0)$ until either the boundary $(\partial Inv(l_0))$ is hit at time $t_*(x_0)$ (which is defined similarly as t_* for the PDP) or until a point is generated by a Poisson process of one of the spontaneous transitions. For each spontaneous transition $\alpha = (l_0, \lambda_\alpha, l', R_\alpha)$ a Poisson process is 'running' with density $\lambda_\alpha(x_t)$. As soon as one of these Poisson processes generates a point, the corresponding spontaneous transition will be taken. If the first jump is caused by a boundary-hit at boundary point ζ, a boundary hit transition will be selected according to the probability measure $C(l_0, \zeta)$. The new continuous state in the target location of the active transition, will be selected according to the probability measures of the reset map R of the boundary hit transition. If the first jump is caused by one of the Poisson processes, the reset map of the corresponding spontaneous transition will select the new continuous state in the target location. From the new hybrid state on, this execution procedure can be repeated.

4 Composition of CPDPs

In this section we define a composition operator for CPDPs. We prove that, under certain conditions, the class of CPDPs is closed under this composition operation. We also prove that the composition operator is commutative and associative. For an explanation of the active/passive communication mechanism, established by this composition operator, we refer to [3].

Suppose CPDPs $\mathcal{A}_i = (L_i, V_i, v_i, Inv_i, G_i, \Sigma, B_i, P_i, S_i, C_i)$ are given. We assume that the sets of communication labels are the same for $\mathcal{A}_1$ and $\mathcal{A}_2$ and we assume that V_1 and V_2 are disjoint. The composition $\mathcal{A}_1||\mathcal{A}_2$ of $\mathcal{A}_1$ with $\mathcal{A}_2$ is defined as follows:

$\mathcal{A}_1||\mathcal{A}_2 := (L, V, v, Inv, G, \Sigma, B, P, S, C)$, where $L = L_1 \times L_2$, $V = V_1 \cup V_2$, $v(l_1, l_2) := v(l_1) \cup v(l_2)$, $Inv((l_1, l_2), y) = Inv_1(l_1, y)$ if $y \in V_1$ and $Inv((l_1, l_2), y) = Inv_2(l_2, y)$ if $y \in V_2$, $G((l_1, l_2), y) = G_1(l_1, y)$ if $y \in V_1$ and $G((l_1, l_2), y) = G_2(l_2, y)$ if $y \in V_2$. The sets B,P and S are determined by the following structural operational rules, where $l_1, l_1' \in L_1$ and $l_2, l_2' \in L_2$. For the boundary hit transitions we have the rules

$$\textbf{r1.}\ \frac{l_1 \xrightarrow{a,R_1} l_1', l_2 \not\xrightarrow{\bar{a}}}{(l_1, l_2) \xrightarrow{a,R} (l_1', l_2)}, \textbf{r2.}\ \frac{l_1 \xrightarrow{a,R_1} l_1', l_2 \xrightarrow{\bar{a},R_2} l_2'}{(l_1, l_2) \xrightarrow{a,R} (l_1', l_2')}$$

These rules should be interpreted as, **r1**: If $(l_1, a, l_1', R) \in B_1$ and there exist no l_2' and R_2 such that $(l_2, \bar{a}, l_2', R_2) \in P_2$, then $((l_1, l_2), a, (l_1', l_2), R) \in B$ (R will be defined next). **r2**: If $(l_1, a, l_1', R) \in B_1$ and $(l_2, \bar{a}, l_2', R_2) \in P_2$, then $((l_1, l_2), a, (l_1', l_2'), R) \in B$. The rules **r3** till **r6** should be interpreted likewise. R in rule **r1** equals R_1 for the variables of l_1' (and thus ignores the valuation of the variables of l_2 before the jump) and equals the 'identity' map for the variables in l_2 (i.e. the values of the variables of l_2 do not change with probability one). R in rule **r2** equals R_1 for the variables of l_1' and equals R_2 for the variables of l_2'. For the spontaneous transitions we have the rules

$$\textbf{r3.}\ \frac{l_1 \xrightarrow{a,R_1,\lambda_1} l_1', l_2 \not\xrightarrow{\bar{a}}}{(l_1, l_2) \xrightarrow{a,R,\lambda} (l_1', l_2)}, \textbf{r4.}\ \frac{l_1 \xrightarrow{a,R_1,\lambda_1} l_1', l_2 \xrightarrow{\bar{a},R_2} l_2'}{(l_1, l_2) \xrightarrow{a,R,\lambda} (l_1', l_2')},$$

where R in rule **r3** is derived from R_1 as in rule **r1** and R in rule **r4** is derived from R_1 and R_2 as in rule **r2**. For the passive transitions we have the rules

$$\textbf{r5.}\ \frac{l_1 \xrightarrow{\bar{a},R_1} l_1', l_2 \not\xrightarrow{\bar{a}}}{(l_1, l_2) \xrightarrow{\bar{a},R} (l_1', l_2)}, \textbf{r6.}\ \frac{l_1 \xrightarrow{\bar{a},R_1} l_1', l_2 \xrightarrow{\bar{a},R_2} l_2'}{(l_1, l_2) \xrightarrow{\bar{a},R} (l_1', l_2')},$$

where R in rule **r5** is derived from R_1 as in rule **r1** and R in rule **r6** is derived from R_1 and R_2 as in rule **r2**.

The reset maps of the boundary hit transitions (as a result of rules **r1** and **r2**) are defined well for boundary points where the variables of the second location l_2 are in the interior of the invariant of l_2. However, for 'double boundary points', i.e. for boundary points where both the variables of the first location and the variables of the second location are on the boundaries of the invariants (of l_1 and l_2 respectively), the reset map is ill-defined because the target continuous state is again a boundary state, which is not allowed for CPDPs. For now, we say that the reset maps for these double boundary points are undefined.

Beside the rules **r1** till **r6**, there are also the rules **r1'** till **r5'** which are the mirrored versions of **r1** till **r5**. This means that

$$\textbf{r1'.}\ \frac{l_1 \not\xrightarrow{\bar{a}}, l_2 \xrightarrow{a,R_2} l_2'}{(l_1, l_2) \xrightarrow{a,R} (l_1, l_2')}, \textbf{r2'.}\ \frac{l_1 \xrightarrow{\bar{a},R_1} l_1', l_2 \xrightarrow{a,R_2} l_2'}{(l_1, l_2) \xrightarrow{a,R} (l_1', l_2')},$$

etc. For active transitions, the choice function C is defined as follows: If $\alpha \in B$ is derived from an active transition $\alpha_1 \in B_1$ (via rule **r1** or **r2**), then $C((l_1, l_2), (\zeta_1, \zeta_2))(\alpha)$ equals $C(l_1, \zeta_1)(\alpha_1)$ (in case **r1**) and $C(l_1, \zeta_1)$ (α_1) $C(l_2, \zeta_2, \bar{a})$ (α_2) (in case **r2** with passive transition α_2) for ζ_1 a boundary point and ζ_2 an interior point, equals zero for ζ_1 an interior point and ζ_2 a boundary point, and is 'undefined' for ζ_1 and ζ_2 both boundary points. For the case that $\alpha \in B$ is derived from an active transition $\alpha_2 \in B_2$ (via rule **r1'** or **r2'**), $C((l_1, l_2), (\zeta_1, \zeta_2))(\alpha)$ is defined vice versa. For passive transitions, the choice function C is defined as follows: If $\alpha \in P$ with label $\bar{a}$ is derived from a passive transition $\alpha_1 \in P_1$ (via rule **r5** or **r6**), then $C((l_1, l_2), (\zeta_1, \zeta_2))(\alpha)$ equals $C(l_1, \zeta_1, \bar{a})(\alpha_1)$ (in case **r5**) and $C(l_1, \zeta_1, \bar{a})(\alpha_1)C(l_2, \zeta_2, \bar{a})(\alpha_2)$ (in case **r6** with passive transition α_2) for ζ_1 and ζ_2 interior points. This ends the definition of composition of CPDPs.

In the definition of composition above, reset maps and choice function are not defined for double boundary points. If our model would allow non-determinism and the possibility to jump onto the boundary (like the more general CPDP model of [13]), we expect that this 'problem' can be solved in a more satisfactory way.

Theorem 1. *The composition of two CPDPs is a CPDP that is undefined on double boundary points assumed that there is no zeno-behavior. With other words, if for the composition of two CPDPs we assign proper reset maps to the double boundary points for the boundary hit transitions and properly define the choice function for the double boundary points, then the composition is a CPDP assumed that this completed composition is non-zeno.*

Proof. It can directly be seen that the elements L,V,v,Inv and G are proper CPDP elements. It can also easily be seen that the transitions that are generated by the rules **r1** till **r6** (and their mirror rules) have proper reset maps (except on the double boundary points) and are therefore proper CPDP transitions (except on the double boundary points). The only element that needs a closer look is the choice function C. For C to be a proper CPDP element, for each boundary point the values that C assigns to the boundary hit transitions should add up to one and also for each interior point (l, ζ) and each passive label $\bar{a}$ that is used by at least one transition of location l, the values that C assigns to the passive transitions in l with label $\bar{a}$ should add up to one. Concerning the active transitions: At a boundary point $(l_1, \partial\zeta_1, l_2, \zeta_2)$, with $\partial\zeta_1 \in \partial Inv_1(l_1)$ and $\zeta_2 \in Inv_2(l_2)$, the value of any active transition α of $\mathcal{A}_1$ with label a is carried over to the corresponding active transition in $\mathcal{A}$ in case that $l_2 \stackrel{\bar{a}}{\not\rightarrow}$ and in case that $l_2 \stackrel{\bar{a}}{\rightarrow}$, this value is spread over the different active transitions that are the result of α synchronizing with the passive $\bar{a}$-transitions in l_2 (i.e. we get $C(l_1, \partial\zeta_1)(\alpha_1)C(l_2, \zeta_2, \bar{a})(\tilde{\alpha}_1) + \cdots + C(l_1, \partial\zeta_1)(\alpha_1)C(l_2, \zeta_2, \bar{a})(\tilde{\alpha}_n) = C(l_1, \partial\zeta_1)(\alpha_1)$, with $\tilde{\alpha}_i$ the passive $\bar{a}$-transitions from l_2). Therefore, because the active transitions corresponding to active transitions in l_2 get value zero, the values add up to one. For boundary points $(l_1, \zeta_1, l_2, \partial\zeta_2)$ we have the symmetric case. For boundary points $(l_1, \partial\zeta_1, l_2, \partial\zeta_2)$, C is undefined. Concerning the passive transitions: With a similar argument it can be shown that values of passive transitions of $\mathcal{A}_i$ either carry over to passive transitions of $\mathcal{A}$ or are spread over a set of passive transitions of $\mathcal{A}$ such that the sum of the values does not change. This ends the proof.

Remark 2. In the composition of CPDPs $\mathcal{A}_1$ and $\mathcal{A}_2$ we get for each joint location (l_1, l_2) a combination of vectorfields from $\mathcal{A}_1$ and $\mathcal{A}_1$. In order to maintain the PDP properties, this composition of vectorfields should be locally Lipschitz continuous. We also get a composition of reset-maps which should result in proper reset maps etc. Because the CPDP is now, as opposed to [3], defined as having multiple variables in one location, these 'properties maintained in composition' are already proved in the PDP/CPDP-equivalence proof from [4].

Corollary 1. *If the probability that two CPDPs (which are composed with each other) reach their boundaries at the same time is zero, then the stochastic behavior of the composite system is fully specified and is equal to the behavior of a PDP. Thus, if we then complete the composition of these two CPDPs to form a new CPDP (which can always be done) in two different ways, then the stochastic behaviors of these two completed CPDPs will be the same.*

Theorem 2. *The composition operator $||$, which operates on the class of CPDPs, is commutative and associative.*

Proof. We identify joint locations (l_1, l_2) of $\mathcal{A}_1||\mathcal{A}_2$ with joint locations (l_2, l_1) of $\mathcal{A}_2||\mathcal{A}_1$. It can directly be seen that the elements L,V,v,Inv and G cause no problems for commutativity and associativity. That the active/passive operator $||$ generates the same transitions for $\mathcal{A}_1||\mathcal{A}_2$ as for $\mathcal{A}_2||\mathcal{A}_1$ and generates the same transitions for $(\mathcal{A}_1||\mathcal{A}_2)||\mathcal{A}_3$ as for $\mathcal{A}_1||(\mathcal{A}_2||\mathcal{A}_3)$ is proven in the case of labelled transition systems in [14]. This result can easily be generalized to the case of CPDPs.

5 Bisimulation for PDPs

In this section we introduce a notion of bisimulation for weighted PDPs (i.e. PDPs together with a weight-function on the state space). Briefly said, two PDP states x and y (in two different PDPs) are bisimilar if first, their piecewise deterministic paths simulate each other (i.e. produce the same weight value for each time instant). If second, at any time instant the states of the paths are again bisimilar. If third, the jump intensities at states x and y are equal. If fourth, the transition measures $Q(x)$ and $Q(y)$ are equivalent probability measures. (The notion of equivalent measures will be defined below).

The state space of a PDP as defined in [2] is a standard Borel space. A measurable space $(E, \mathcal{E})$, with $\mathcal{E}$ the Borel sets of E, is called a standard Borel space, if E is homeomorphic to a Borel subset of a complete separable metric space. In order to prove stochastic equivalence of two bisimilar PDPs, we will need that the quotient spaces (induced by a bisimulation relation) are also standard Borel spaces.

We define the equivalence relation on X that is induced by a relation $\mathcal{R} \subset X \times Y$ with the property that $\pi_1(\mathcal{R}) = X$ and $\pi_2(\mathcal{R}) = Y$ as the transitive closure of $\{(x, x')|\exists y \text{ s.t. } (x, y) \in \mathcal{R} \text{ and } (x', y) \in \mathcal{R}\}$. We write $X/_\mathcal{R}$ and $Y/_\mathcal{R}$ for the sets of equivalence classes of X and Y induced by $\mathcal{R}$. We denote the equivalence class of $x \in X$ by $[x]$. We will now define the notion of *measurable relations* and of *equivalent measures*, which we need for our notion of bisimulation for PDPs.

Definition 1. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be standard Borel spaces and let $\mathcal{R} \subset X \times Y$ be a relation such that $\pi_1(\mathcal{R}) = X$ and $\pi_2(\mathcal{R}) = Y$. Let $\mathcal{X}^*$ be the collection of all $\mathcal{R}$-saturated Borel sets of X, i.e. all $B \in \mathcal{X}$ such that any equivalence class of X is either totally contained or totally not contained in B. It can be checked that $\mathcal{X}^*$ is a σ-algebra. Let*

$$\mathcal{X}^*/_\mathcal{R} = \{[A]|A \in \mathcal{X}^*\},$$

where $[A] := \{[a]|a \in A\}$. Then $(X/_\mathcal{R}, \mathcal{X}^*/_\mathcal{R})$, which is a measurable space, is called the quotient space *of X with respect to $\mathcal{R}$. A unique bijective mapping $f : X/_\mathcal{R} \to Y/_\mathcal{R}$ exists, such that $f([x]) = [y]$ if $(x, y) \in \mathcal{R}$. We say that the relation $\mathcal{R}$ is* measurable *if for all $A \in \mathcal{X}^*/_\mathcal{R}$ we have $f(A) \in \mathcal{Y}^*/_\mathcal{R}$ and vice versa.*

If a relation on $X \times Y$ is measurable, then the quotient spaces of X and Y are homeomorphic (under bijection f from Definition 1). We could say therefore that under a measurable relation X and Y have a shared quotient space. In the field of descriptive set theory, a relation $\mathcal{R} \subset X \times Y$ is called measurable if $\mathcal{R} \in \mathcal{B}(X \times Y)$ (i.e. $\mathcal{R}$ is a Borel set of the space $X \times Y$). This definition does not coincide with our definition of measurable relation. In fact, many interesting measurable relations are not Borel sets of the product space $X \times Y$.

Definition 2. *Suppose we have measures P_X and P_Y on standard Borel spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ respectively. Suppose that we have a measurable relation $\mathcal{R} \subset X \times Y$. The measures P_X and P_Y are called equivalent with respect to $\mathcal{R}$ if we have $P_X(f_X^{-1}(A)) = P_Y(f_Y^{-1}(f(A)))$ for all $A \in \mathcal{X}^*/_\mathcal{R}$ (with f as in Definition 1 and with f_X and f_Y the mappings that map X and Y to $X/_\mathcal{R}$ and $Y/_\mathcal{R}$ respectively).*

Suppose we have a PDP with state-space X and $weight_X$ is a real-valued measurable function on X. Then we call the PDP together with $weight_X$ a *weighted PDP*. The function $weight_X$ can be seen as a weight function on the state-space. It can also be seen as an output at the state or as the observable component. We call $weight_X$ the weight-function or the output-function. We will now define a bisimulation notion for weighted PDPs. In this definition we write $Q(x)$ (or $Q(y)$) for the reset map of the PDP with state space X (or Y) at state x (or y). We write $\phi(t, x)$ for the value of the state at time t when the PDP with state space X starts at x at $t = 0$, etc.

Definition 3. *Suppose we have two weighted PDPs with state-spaces X and Y and weight-functions $weight_X$ and $weight_Y$. A measurable relation $\mathcal{R} \subset X \times Y$ is a* bisimulation *iff $(x, y) \in \mathcal{R}$ implies that*

- *$weight_X(x) = weight_Y(y)$, $t_*(x) = t_*(y)$ and $\lambda(x) = \lambda(y)$.*
- *$(\phi(t,x), \phi(t,y)) \in \mathcal{R}$ for all $t \in [0, t_*(x)[$.*
- *$Q(x)$ and $Q(y)$ are equivalent probability measures with respect to $\mathcal{R}$. Also $Q(\phi(t_*(x),x))$ and $Q(\phi(t_*(y),y))$ are equivalent probability measures with respect to $\mathcal{R}$.*

Two states x and y are bisimilar if they are contained in some bisimulation.

The following theorem shows that bisimilar PDPs exhibit equivalent stochastic behavior. We make use of the *Hilbert cube* probability space, which has as sample space $\Omega = \prod_{i=1}^{\infty} Y_i$, where each $Y_i = [0,1]$, and has the product Borel sigma-algebra and product Lebesgue measure.

Theorem 3. *If initial states x and y of two weighted PDPs $(X, weight_X)$ and $(Y, weight_Y)$ are contained in bisimulation $\mathcal{R}$, then, assumed that the quotient spaces are standard Borel spaces, we can construct the stochastic processes x_t and y_t on the Hilbert cube $(\Omega, \mathcal{A}, P)$ in such a way that for each $\omega \in \Omega$ we have $weight_X(x_t(\omega)) = weight_Y(y_t(\omega))$.*

Proof. Let $\mathcal{R} \subset X \times Y$ be a bisimulation such that $(x,y) \in \mathcal{R}$. Let $(\Omega, \mathcal{A}, P)$ be the Hilbert cube and $U_i(\omega) = \omega_i$ be the $U[0,1]$ distributed random variables. We define for any z that has a corresponding survivor function $F(t,z)$

$$\psi_1(u,z) = \begin{cases} \inf\{t | F(t,z) \leq u\} \\ +\infty \text{ if the above set is empty} \end{cases}$$

We define the random variables $S_{1,x}$, $T_{1,x}$, $S_{1,y}$ and $T_{1,y}$ as $S_{1,x}(\omega) = T_{1,x}(\omega) = \psi_1(U_1(\omega), x)$ and $S_{1,y}(\omega) = T_{1,y}(\omega) = \psi_1(U_1(\omega), y)$. Now we can define the sample-functions up to the first jump. For $z \in \{x,y\}$ we define: if $T_{1,z}(\omega) = \infty$ then $z_t(\omega) = \phi(t,z)$ for $t \geq 0$, if $T_{1,z}(\omega) < \infty$ then $z_t(\omega) = \phi(t,z)$ for $0 \leq t < T_{1,z}(\omega)$.

Because $(x,y) \in \mathcal{R}$, we have $t_*(x) = t_*(y)$ and $(\phi(t,x), \phi(t,y)) \in \mathcal{R}$ for $t \in [0, t_*(x)[$. We also have $\lambda(\phi(t,x)) = \lambda(\phi(t,y))$ for $t \in [0, t_*(x)[$. Now it can be easily checked that $F(t,x) = F(t,y)$ for all $t \in \mathbb{R}$ and therefore $\psi(u,x) = \psi(u,y)$ and we have $S_{1,x}(\omega) = S_{1,y}(\omega)$ and $T_{1,x}(\omega) = T_{1,y}(\omega)$. Because $(\phi(t,x), \phi(t,y)) \in \mathcal{R}$ we have $weight_X(x_t(\omega)) = weight_Y(y_t(\omega))$ up to $T_{1,x}(\omega)$.

Now $x_{T_1}(\omega)$ and $y_{T_1}(\omega)$, where $T_1 := T_{1,x}(\omega) = T_{1,y}(\omega)$, need to be chosen in accordance to $Q(\phi(T_1,x))$ and $Q(\phi(T_1,y))$ respectively. Because $(x,y) \in \mathcal{R}$, we have that $Q' := Q(\phi(T_1,x))$ and $Q'' := Q(\phi(T_1,y))$ are equivalent probability measures with respect to $\mathcal{R}$. Therefore, Q' and Q'' define the same probability measure P_Z on the quotient space $(Z, \mathcal{Z})$. Let $P_{X|z}$ and $P_{Y|z}$ be the conditional probability measures of Q' and Q'' given the outcome z in Z. Because X, Y and Z are all separable standard Borel spaces, these conditional probability measures exist uniquely according to Th.8.1 in [15] and according to the same theorem we have that for fixed $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ the maps $z \rightarrow P_{X|z}(A)$ and $z \rightarrow P_{Y|z}(B)$ are measurable.

Let $\psi_2 : [0,1] \times X \cup \partial X \to Z$ be a measurable mapping such that $l\psi_2^{-1}(A,x) = P_Z(A,x)$ for all $x \in X \cup \partial X$. The existence of this mapping follows from Corollary 23.4 in [2] and from the fact that the mapping $x \to P_Z(A,x)$ is measurable for fixed $A \in \mathcal{Z}$. Let $\psi_{3,x} : [0,1] \times Z \to X$ and $\psi_{3,y} : [0,1] \times Z \to Y$ be measurable mappings such that $l\psi_{3,x}^{-1}(A,z) = P_{X|z}(A)$ for $A \in \mathcal{B}(X)$ and $l\psi_{3,y}^{-1}(A,z) = P_{Y|z}(A)$ for $A \in \mathcal{B}(Y)$. The existence of these mappings follows from Corollary 23.4 in [2] and from the fact that for fixed $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ the mappings $z \to P_{X|z}(A)$ and $z \to P_{Y|z}(B)$ are measurable. Now the processes x_t and y_t restart at time $T_1(\omega)$ from the states $x_{T_1}(\omega) = \psi_{3,x}(U_3(\omega), \psi_2(U_2(\omega), \phi(T_1(\omega), x)))$ and $y_{T_1}(\omega) = \psi_{3,y}(U_3(\omega), \psi_2(U_2(\omega), \phi(T_1(\omega), x)))$ and we have $(x_{T_1}(\omega), y_{T_1}(\omega)) \in \bar{\mathcal{R}}$, where $\bar{\mathcal{R}}$ is defined as $\{(x,y)| f([x]) = [y]\}$ (see Definition 1). To continue the sample function from time $T_1(\omega)$, we define $S_{2,x} = \psi_1(U_4(\omega), x_{T_1}(\omega))$, $S_{2,y} = \psi_1(U_4(\omega), y_{T_1}(\omega))$, $T_{2,x} = T_{1,x}(\omega) + S_{2,x}(\omega)$, $T_{2,y} = T_{1,y}(\omega) + S_{2,y}(\omega)$, and we repeat the procedure above.

It can be seen that the stochastic processes above are well defined and that for all $t \geq 0$ and all $\omega \in \Omega$ we have $(x_t(\omega), y_t(\omega)) \in \bar{\mathcal{R}}$. This means that $weight_X(x_t(\omega)) = weight_Y(y_t(\omega))$. This ends the proof.

Corollary 2. *Because $weight_X$ and $weight_Y$ in Proposition 3 are measurable mappings, we have that $z_t = weight_X(x_t)$ and $z'_t = weight_Y(y_t)$ are well-defined stochastic processes. Because $weight_X(x_t(\omega)) = weight_Y(y_t(\omega))$, the stochastic processes z_t and z'_t are indistinguishable. Thus, if two weighted PDPs have bisimilar initial states (and the quotient spaces are standard Borel spaces) then there is a realization of the stochastic processes of their outputs (on the Hilbert cube) such that the stochastic processes are indistinguishable .*

Remark 3. For sake of simplicity we assumed that weight-functions take value in $\mathbb{R}$. However, all results still hold if we take any other euclidean space than $\mathbb{R}$ as codomain of the weight functions.

6 Bisimulation for CPDPs

We will now generalize the notion of bisimulation for PDPs to CPDPs. To do that, we need to introduce the concept of weighted CPDPs.

Definition 4. *A weighted CPDP is a CPDP together with a set of output variables $W = \{w_1, w_2, \cdots, w_n\}$, where each w_i takes value in $\mathbb{R}^{d(w_i)}$, with $d(w_i)$ the dimension of w_i, and an output function weight which assigns to each $w \in W$ and each CPDP state x a value $weight(w,x) \in \mathbb{R}^{d(w)}$. weight is such that for fixed w the functions $weight(w,x)$ are measurable.*

For composition of two CPDPs with state spaces X_1 and X_2, with disjoint sets of output variables W_1 and W_2 and with output functions $weight_1$ and $weight_2$, the composed output function $weight$ assigns to $(w,(x_1,x_2))$ the value $weight_1(w,x_1)$ if $w \in W_1$ and $weight_2(w,x_2)$ if $w \in W_2$. In order to define bisimulation for CPDPs we also need to introduce the notions of *combined reset map* and *combined jump rate function*:

For CPDP $\mathcal{A} = (L, V, v, Inv, G, \Sigma, B, P, S, C)$ with hybrid state space E, We define R, which we call the combined reset map, as follows. R assigns to each triplet (l, ζ, a) with $(l, \zeta) \in \partial E$ and with $a \in \Sigma$ such that $l \xrightarrow{a}$ (i.e. there exists a boundary hit transition labelled a leaving l), a measure on E. This measure $R(l, \zeta, a)$ is for any l' and any Borel set $A \subset Inv_{l'}$ defined as:

$$R(l, \zeta, a)(A) = \sum_{\alpha \in B_{l,a,l'}} C(l, \zeta)(\alpha) R_\alpha(A),$$

where $B_{l,a,l'}$ denotes the set of boundary hit transitions from l to l' with label a. (This measure is uniquely extended to all Borel sets of E). Now, for $A \in \mathcal{B}(E)$, $R(l, \zeta, a)(A)$ equals the probability of jumping into A via a boundary hit transition with label a given that the jump takes place at (l, ζ). Furthermore, R assigns to each triplet $(l, \zeta, \bar{a})$ with $(l, \zeta) \in E$ and with $a \in \bar{\Sigma}$ such that $l \xrightarrow{\bar{a}}$, a measure on E, which for any l' and any Borel set $A \subset Inv_{l'}$ is defined as:

$$R(l, \zeta, \bar{a})(A) = \sum_{\alpha \in P_{l,\bar{a},l'}} C(l, \zeta, \bar{a})(\alpha) R_\alpha(A).$$

(This measure is uniquely extended to all Borel sets of E). Now, $R(l, \zeta, \bar{a})(A)$, with $A \in \mathcal{B}(E)$, equals the probability of jumping into A if a passive transition with label $\bar{a}$ takes place at (l, ζ). We define the combined jump rate function λ for CPDP $\mathcal{A}$ as

$$\lambda(l, \zeta) = \sum_{\alpha \in S_{l \to}} \lambda_\alpha,$$

with $(l, \zeta) \in E$. Finally, for spontaneous jumps, R assigns to each $(l, \zeta) \in E$ such that $\lambda(l, \zeta) \neq 0$, a probability measure on E, which for any l' and any Borel set $A \subset Inv_{l'}$ is defined as:

$$R(l, \zeta)(A) = \sum_{\alpha \in S_{l \to l'}} \frac{\lambda_\alpha(l, \zeta)}{\lambda(l, \zeta)} R_\alpha(A).$$

Now we are ready to give the definition of bisimulation for CPDPs.

Definition 5. *Suppose we have two weighted CPDPs with state-spaces X and Y and weight-functions $weight_X$ and $weight_Y$ on a shared set of output variables W. A measurable relation $\mathcal{R} \subset X \times Y$ is a* bisimulation *iff $(x, y) \in \mathcal{R}$, with $x = (l_1, \zeta_1)$ and $y = (l_2, \zeta_2)$, implies that*

- *$weight_X(w, x) = weight_Y(w, y)$ for all $w \in W$, $t_*(x) = t_*(y)$ and $\lambda(x) = \lambda(y)$.*
- *$(\phi(t, x), \phi(t, y)) \in \mathcal{R}$ for all $t \in [0, t_*(x)[$.*
- *If $\lambda(x) = \lambda(y) \neq 0$, then $R(x)$ and $R(y)$ are equivalent probability measures with respect to $\mathcal{R}$. For any $\bar{a} \in \bar{\Sigma}$ we have that either both $l_1 \stackrel{\bar{a}}{\not\longrightarrow}$ and $l_2 \stackrel{\bar{a}}{\not\longrightarrow}$*

or else $R(x, \bar{a})$ *and* $R(y, \bar{a})$ *are equivalent probability measures. Also, if we define* $(l_1, \zeta_1^*) := t_*(x)$ *and* $(l_2, \zeta_2^*) := t_*(y)$*, then we have for any* $a \in \Sigma$ *that either both* $l_1 \stackrel{a}{\not\longrightarrow}$ *and* $l_2 \stackrel{a}{\not\longrightarrow}$ *or else* $R(l_1, \zeta_1^*, a)$ *and* $R(l_2, \zeta_2^*, a)$ *are equivalent measures.*

Two states x *and* y *are bisimilar if they are contained in some bisimulation.*

Theorem 4. *The stochastic processes of the outputs of two bisimilar closed CPDPs, whose quotient spaces are standard Borel spaces, can be realized such that they are indistinguishable.*

Proof. The stochastic process of a closed CPDP $\mathcal{A}$ with combined reset map R and combined jump rate function λ is equivalent (i.e. indistinguishable) with the stochastic process of the PDP $\tilde{\mathcal{A}}$ that has the same state space and vectorfields as the CPDP and that has λ as its jump rate function and has transition measure $Q(l, \zeta)$ that equals $R(l, \zeta)$ for interior points and that equals $\sum_{a \in \Sigma} R(l, \zeta, a)$ for boundary points (see [4] for the proof of this stochastic equivalence). This PDP $\tilde{\mathcal{A}}$ is called the corresponding PDP of CPDP $\mathcal{A}$. We prove that the corresponding PDPs of two bisimilar closed CPDPs are bisimilar PDPs, then the result follows from Corollary 2:

The first two lines of Definition 3 follow directly from the first two lines of Definition 5. The third line: The fact that $Q(x)$ and $Q(y)$ are equivalent probability measures for bisimilar interior points x and y follows from the fact that $Q(x) = R(x)$ and $Q(y) = R(y)$ and, according to Definition 5, $R(x)$ and $R(y)$ are equivalent probability measures. Finally, $Q(\phi(t_*(x), x))$ and $Q(\phi(t_*(y), y))$ are equivalent probability measures because $Q(\phi(t_*(x), x)) = \sum_{a \in \Sigma} R(\phi(t_*(x), x), a)$ and $Q(\phi(t_*(y), y)) = \sum_{a \in \Sigma} R(\phi(t_*(y), y), a)$ and, according to Definition 5, $R(\phi(t_*(x), x), a)$ and $R(\phi(t_*(x), x), a)$ are equivalent measures for all $a \in \Sigma$. This ends the proof.

In order to prove the main theorem 5 about bisimulation in the context of composition, we need that a measurable relation $\mathcal{R} \subset X_1 \times X_2$ naturally induces a measurable relation $\mathcal{R}'$ on $(X_1 \times Y) \times (X_2 \times Y)$ for any Y. This result is proved in the following lemma. After that, the main theorem is stated.

Lemma 1. *If* $\mathcal{R} \subset X_1 \times X_2$ *is a measurable relation such that* $\pi_1(\mathcal{R}) = X_1$, $\pi_2(\mathcal{R}) = X_2$ *and* $X_1/_{\mathcal{R}}$ *and* $X_2/_{\mathcal{R}}$ *are standard Borel spaces, then*

$$\mathcal{R}' := \{((x_1, y), (x_2, y)) | (x_1, x_2) \in \mathcal{R}, y \in Y\}$$

is a measurable relation on $(X_1 \times Y) \times (X_2 \times Y)$ *and* $(X_1 \times Y)/_{\mathcal{R}'}$ *and* $(X_2 \times Y)/_{\mathcal{R}'}$ *are standard Borel spaces.*

Proof. From the proof of Theorem 3 we know that (because $X_1/_{\mathcal{R}}$ is a standard Borel space) there exists a measurable $\psi : [0,1] \times X_1/_{\mathcal{R}} \to X_1$, such that $\psi([0,1] \times [x]) = \{\tilde{x} \in X_1 | [\tilde{x}] = [x]\}$. We first proof that $(X_1 \times Y)/_{\mathcal{R}'} = X_1/_{\mathcal{R}} \times Y$, which is indeed a standard Borel space.

Take $B \in \mathcal{B}^*(X_1 \times Y)$ (i.e. B is Borel in $X_1 \times Y$ and for any y we have: if $(x, y) \in B$ and $[x] = [\tilde{x}]$ then $(\tilde{x}, y) \in B$). Now there exist Borel sets $B_i^{X_1}$ and B_i^Y such that

$$B = \cup_{i=1}^{\infty} B_i^{X_1} \times B_i^Y.$$

Because ψ is measurable, we have that for all i that $\psi^{-1}(B_i^{X_1}) \in \mathcal{B}([0,1] \times X_1/_{\mathcal{R}})$. This means that there exist Borel sets $B_{i,j}^{X_1/\mathcal{R}}$ and $B_{i,j}^{[0,1]}$, such that

$$\cup_{i=1}^{\infty} \psi^{-1}(B_i^{X_1}) \times B_i^Y = \cup_{i,j=1}^{\infty} B_{i,j}^{[0,1]} \times B_{i,j}^{X_1/\mathcal{R}} \times B_i^Y$$

Because we have that if $(x, y) \in B$ and $[x] = [\tilde{x}]$ then $(\tilde{x}, y) \in B$), we can also write

$$\cup_{i=1}^{\infty} \psi^{-1}(B_i^{X_1}) \times B_i^Y = \cup_{i,j=1}^{\infty} [0,1] \times B_{i,j}^{X_1/\mathcal{R}} \times B_i^Y,$$

from which we can see that $\mathcal{R}'$ maps B to $\cup_{i,j=1}^{\infty} B_{i,j}^{X_1/\mathcal{R}} \times B_i^Y$, which is a Borel set in $X_1/_{\mathcal{R}} \times Y$ and therefore $(X_1 \times Y)/_{\mathcal{R}'}$ is a standard Borel space. Analogously we get that $\mathcal{R}'$ maps $B \in \mathcal{B}^*(X_2 \times Y)$ to Borel sets in $X_2/_{\mathcal{R}} \times Y$. Measurability of $\mathcal{R}'$ can now, with the results above, easily be derived from the measurability of $\mathcal{R}$. This ends the proof.

Theorem 5. *Suppose we have three weighted CPDPs with state spaces X_1,X_2 and Y, and with output functions $weight_{X_1}$ on W_X, $weight_{X_2}$ on W_X and $weight_Y$ on W_Y respectively. Suppose $\mathcal{R} \subset X_1 \times X_2$ is a bisimulation and $X_1/_{\mathcal{R}}$ and $X_2/_{\mathcal{R}}$ are standard Borel spaces. Then,*

$$\mathcal{R}' := \{((x_1, y), (x_2, y)) | (x_1, x_2) \in \mathcal{R}, y \in Y\}$$

is a bisimulation on $(X_1 \times Y) \times (X_2 \times Y)$ and $(X_1 \times Y)/_{\mathcal{R}'}$ and $(X_2 \times Y)/_{\mathcal{R}'}$ are standard Borel spaces.

Proof. Suppose $((x_1, y), (x_2, y)) \in \mathcal{R}'$ with $x_1 = (l_1, \zeta_1)$, $x_2 = (l_2, \zeta_2)$ and $y = (l_y, \zeta_y)$. We have to prove the three lines of Definition 5 to be true.

First line: For $w \in W_X$, $weight_{X_1||Y}(w, x_1, y) = weight_{X_1}(w, x_1) = weight_{X_2}(w, x_1) = weight_{X_2||Y}(w, x_1, y)$ and for $w \in W_Y$, $weight_{X_1||Y}(w, x_1, y) = weight_Y(w, y) = weight_{X_2||Y}(w, x_1, y)$. $t_*(x_1, y) = \min\{t_*(x_1), t_*(y)\} = \min\{t_*(x_2), t_*(y)\} = t_*(x_2, y)$. $\lambda(x_1, y) = \lambda(x_1) + \lambda(y) = \lambda(x_2) + \lambda(y) = \lambda(x_2, y)$.

Second line: The flow ϕ from states (x_1, y) and (x_2, y) consists of two parts, the x-part: $\phi(t, x_1)$ and $\phi(t, x_2)$, and the y-part: $\phi(t, y)$. The x-part and y-part flows are evolving independently. Then it follows from the fact that $(\phi(t, x_1), \phi(t, x_2)) \in \mathcal{R}$ for all $t \in [0, t_*(x_1, y)[$ that $(\phi(t, x_1, y), \phi(t, x_2, y)) \in \mathcal{R}'$ for $t \in [0, t_*(x_1, y)[$.

Third line (part one): Suppose $\lambda(x_1, y) \neq 0$ (and consequently $\lambda(x_2, y) \neq 0$). Take arbitrary $A_1 \in \mathcal{B}^*(X_1)$ and $B \in \mathcal{B}(Y)$. Let A_2 be the element of $\mathcal{B}^*(X_2)$ that corresponds (according to $\mathcal{R}$) to A_1. Then $A_1 \times B \in \mathcal{B}^*(X_1 \times Y)$ and $A_2 \times B \in \mathcal{B}^*(X_2 \times Y)$. Furthermore, $A_1 \times B$ and $A_2 \times B$ correspond with each other (according to $\mathcal{R}'$). It can be seen that $R(x_1, y)(A_1 \times B) = \frac{\lambda(x_1)}{\lambda(x_1)+\lambda(y)} R(x_1)(A_1) +$

$\frac{\lambda(y)}{\lambda(x_1)+\lambda(y)}R(y)(B) = \frac{\lambda(x_2)}{\lambda(x_2)+\lambda(y)}R(x_2)(A_2) + \frac{\lambda(y)}{\lambda(x_2)+\lambda(y)}R(y)(B) = R(x_2,y)(A_2 \times B)$. It can be shown for $i \in \{1,2\}$, that the σ-algebra $\mathcal{B}^*(X_i \times Y)$ is generated by the collection of sets of the form $A_i \times B$ with $A_i \in \mathcal{B}^*(X_i)$ and $B \in \mathcal{B}(Y)$. Then it follows that $R(x_1,y)$ and $R(x_2,y)$ are equivalent measures with respect to $\mathcal{R}'$ (under the assumption that $\lambda(x_1,y) \neq 0$).

Third line (part two): It can be seen that if $(l_1,l_y) \xrightarrow{\bar{a}}$, then also $(l_2,l_y) \xrightarrow{\bar{a}}$ (and vice versa). Suppose $(l_1,l_y) \xrightarrow{\bar{a}}$ and $(l_2,l_y) \xrightarrow{\bar{a}}$. Take $A_1 \in \mathcal{B}^*(X_1)$ and $B \in \mathcal{B}(Y)$. Let A_2 be the saturated Borel set of X_2 corresponding to A_1. We distinct three cases: If $l_1 \xrightarrow{\bar{a}}$ and $l_y \xrightarrow{\bar{a}}$ (case 1), then $R(x_1,y,\bar{a})(A_1 \times B) = R(x_1,\bar{a})(A_1)R(y,\bar{a})(B) = R(x_2,\bar{a})(A_2)R(y,\bar{a})(B) = R(x_2,y,\bar{a})(A_2 \times B)$. If $l_1 \xrightarrow{\bar{a}}$ and $l_y \not\xrightarrow{\bar{a}}$ (case 2), then $R(x_1,y,\bar{a})(A_1 \times B) = R(x_1,\bar{a})(A_1)I_y(B) = R(x_2,\bar{a})(A_2)I_y(B) = R(x_2,y,\bar{a})(A_2 \times B)$ (here $I_y(B)$ is the probability measure that equals one if $y \in B$ and zero if $y \notin B$). If $l_1 \not\xrightarrow{\bar{a}}$ and $l_y \xrightarrow{\bar{a}}$ (case 3), then $R(x_1,y,\bar{a})(A_1 \times B) = I_{x_1}(A_1)R(y,\bar{a})(B) = I_{x_2}(A_2)R(y,\bar{a})(B) = R(x_2,y,\bar{a})(A_2 \times B)$. We can now conclude that $R(x_1,y,\bar{a})$ and $R(x_2,y,\bar{a})$ are equivalent probability measures.

Third line (part three): It can be seen that if $(l_1,l_y) \xrightarrow{a}$, then also $(l_2,l_y) \xrightarrow{a}$ (and vice versa). Suppose $(l_1,l_y) \xrightarrow{a}$ and $(l_2,l_y) \xrightarrow{a}$. Take $A_1 \in \mathcal{B}^*(X_1)$ and $B \in \mathcal{B}(Y)$. Let A_2 be the saturated Borel set of X_2 corresponding to A_1. We distinct three cases: If $t_*(x_1) < t_*(y)$ (case 1), then, if we define $x_1^* := \phi(t_*(x_1),x_1)$, $x_2^* := \phi(t_*(x_1),x_2)$ and $y^* := \phi(t_*(x_1),y)$, we get $R(x_1^*,y^*,a)(A_1 \times B) = R(x_1^*,a)(A_1)I_{y^*}(B) = R(x_2^*,a)(A_2)I_{y^*}(B) = R(x_2^*,y^*,a)(A_2 \times B)$. If $t_*(x_1) > t_*(y)$ (case 1), then, if we define $x_1^* := \phi(t_*(y),x_1)$, $x_2^* := \phi(t_*(y),x_2)$ and $y^* := \phi(t_*(y),y)$, we get $R(x_1^*,y^*,a)(A_1 \times B) = I_{x_1^*}(A_1) \times R(y^*,a)(B) = I_{x_2^*}(A_2) \times R(y^*,a)(B) = R(x_2^*,y^*,a)(A_2 \times B)$. If $t_*(x_1) = t_*(y)$ (case 3: double boundary point), then both $R(x_1^*,y^*,a)$ and $R(x_2^*,y^*,a)$ are undefined. We can now conclude that $R(x_1^*,y^*,a)$ and $R(x_2^*,y^*,a)$ are equivalent probability measures. This ends the proof.

Corollary 3. *If a component of a complex CPDP (consisting of multiple CPDPs composed with the composition operator* $||$*) is substituted by a different but bisimilar component, then the stochastic behavior of the complex CPDP will not change.*

We now give two examples of bisimular CPDPs. The examples highlight different aspects of CPDP bisimulation.

Example 1 (State space transformation/reduction). If we have a CPDP which has linear time invariant dynamics $\dot{x} = Ax, weight(x) = Cx$, in (one of) its locations and T is a state space transformation matrix, then the CPDP that is obtained by transforming the dynamics into $\dot{\tilde{x}} = TAT^{-1}\tilde{x}, weight(\tilde{x}) = CT^{-1}\tilde{x}$, is bisimilar to the original CPDP. Classical state space reduction within a CPDP with LTI dynamics also results in a bisimilar CPDP.

Example 2 (Combining Poisson processes). Suppose we have a CPDP which has two spontaneous transitions, with jump rate functions $\lambda_1(x)$ and $\lambda_2(x)$ and reset maps $R_1(x)$ and $R_2(x)$, that have the same label and the same origin and target location. Replacing these two transitions by one spontaneous transition with the same origin and target location and with jump rate function $\lambda_1 + \lambda_2$ and reset map $\frac{\lambda_1}{\lambda_1+\lambda_2}R_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}R_2$, will not change the CPDP up to bisimilarity.

7 Conclusions

We introduced the CPDP model for compositional modelling of PDP systems. We defined a composition operator on CPDPs based on the communication via active and passive events and we defined a notion of bisimulation. We proved that the output processes of closed bisimilar CPDPs are indistinguishable stochastic processes and we also proved that, within a CPDP composition context, substituting a component by another bisimilar component, does not change the system up to bisimilarity. This means that we can use bisimulation as a compositional technique for state reduction. Components that are state-reduced by bisimulation are still of the PDP type and also the composition of these components is still of the PDP type. This means that both the components and the composite system can in principle be analyzed by using PDP analysis techniques (as developed in [2]).

In for example [8] and [10] algorithms are given for finding maximal bismulations for a given system. Since the class of CPDPs is very broad, general algorithms for bisimulation may be difficult to formulate or may not be very useful. Instead, an interesting direction for future research is to define subclasses of CPDPs (such as CPDPs with linear dynamics) that allow development of automatic bisimulation techniques.

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